Research Article

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A Remark on Newest Vertex Bisection in Any Space Dimension

Abstract: With newest vertex bisection, there is no uniform bound on the number of \( n \)-simplices that need to be refined to arrive at the smallest conforming refinement \( T' \) of a conforming partition \( T \) in which one simplex has been bisected. In this note, we show that the difference in levels between any \( T' \in T \) and its ancestor \( T \in T \) is uniformly bounded. This result has been used in [2, Lemma 4.2] by Carstensen and the first two authors.

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1 Newest Vertex Bisection

From [3, 5, 6], we recall the generalization to \( n \geq 2 \) dimensions of the newest vertex bisection algorithm. A tagged simplex \((z_0, \ldots, z_n; \gamma)\) is an \((n+2)\)-tuple of vertices \(z_0, \ldots, z_n \in \mathbb{R}^n\), which do not lie on an \((n-1)\)-dimensional hyperplane, and of a type \(\gamma \in \{0, \ldots, n-1\}\). The mapping \(\text{dom}(\cdot) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times (0, \ldots, n-1) \to 2\mathbb{R}^n\) extracts the corresponding (closed) simplex \(\text{dom}(z_0, \ldots, z_n; \gamma) := \text{conv}(z_0, \ldots, z_n)\) from a tagged simplex \((z_0, \ldots, z_n; \gamma)\). For convenience, for a tagged simplex \(T\) we often denote \(\text{dom}(T)\) simply as \(T\).

The bisection of a tagged simplex \((z_0, \ldots, z_n; \gamma)\) generates the two tagged simplices
\[
\left(\frac{z_0 + z_1}{2}, z_1, \ldots, z_{\gamma+1}, \ldots, z_{n-1}; (\gamma + 1) \mod n\right),
\]
\[
\left(\frac{z_n + z_1}{2}, z_1, \ldots, z_{\gamma-1}, \ldots, z_{n+1}; (\gamma + 1) \mod n\right).
\]

(By convention, the finite sequences \((z_{\gamma+1}, \ldots, z_{n-1})\) and \((z_1, \ldots, z_{\gamma})\) are void for \(\gamma = n-1\) and \(\gamma = 0\), respectively.) The edge \(\text{conv}(z_0, z_n)\) of the original simplex that has been cut is known as its refinement edge. The two new tagged simplices are called the children of the tagged simplex \((z_0, \ldots, z_n; \gamma)\), and any child of some child of a tagged simplex is called grandchild.

Let \(T_0\) be an initial, conforming triangulation of a polyhedral bounded Lipschitz domain \(\Omega \subseteq \mathbb{R}^n\) into tagged \(n\)-simplices. This means that the corresponding set of simplices \(\{T : T \in T_0\}\) covers the domain \(\Omega\), and that two distinct simplices \(T = \text{conv}(y_0, \ldots, y_n)\) and \(T' := \text{conv}(z_0, \ldots, z_n)\) from \(T_0\) are either disjoint or share exactly one surface (e.g., an edge or side) in the sense that there exist \(0 \leq j_1 < \cdots < j_N \leq n\) and \(0 \leq k_1 < \cdots < k_N \leq n\) for some \(N \in \{1, \ldots, n\}\) such that
\[
T \cap T' = \text{conv}(y_{j_1}, \ldots, y_{j_N}) = \text{conv}(z_{k_1}, \ldots, z_{k_N}).
\]

We will exclusively consider partitions of tagged simplices that are descendants of \(T_0\), meaning that they can be created by recurrent bisections of individual simplices in the triangulation starting from \(T_0\). Such partitions are uniformly shape regular in the sense that for any simplex \(T\) from any of these partitions
\[
\text{meas}(T)^{1/n} = \text{diam}(T) = 2^{-\varepsilon(T)/n}
\]
only dependent on \( \mathcal{T}_0 \). Here \( \ell(T) \) denotes the level of \( T \), being the number of bisections that are needed to create \( T \) from a simplex \( T' \) in \( \mathcal{T}_0 \). Note that \( \ell(T) = \text{meas}(T)/\text{meas}(T') \).

Here and in the following, by \( C \leq D \) we will mean that \( C \) can be bounded by a multiple of \( D \), only dependent on the initial triangulation \( \mathcal{T}_0 \). Furthermore, \( C \geq D \) is defined as \( D \leq C \), and \( C = D \) as \( C \leq D \) and \( C \geq D \).

In view of applications in adaptive finite element methods, more specifically we will restrict our considerations to those triangulations that in addition are \textit{conforming}. The set of all \textit{conforming descendants} of \( \mathcal{T}_0 \) will be denoted by \( \mathcal{T} \).

Using the uniform shape regularity and conformity, one easily shows the following result.

**Lemma 1.1.** There exist constants \( C, c > 0 \) such that
(a) for any \( T, T' \in \mathcal{T} \) with \( T \cap T' \neq \emptyset \), it holds that \( |\ell(T) - \ell(T')| \leq C \);
(b) for any \( T, T' \in \mathcal{T} \) with \( \ell(T) > \ell(T') \) + \( C \), it holds that \( \text{dist}(T, T') \geq c 2^{\ell(T)/n} \).

## 2 Matching Condition

Note that, given a tagged simplex \( T = (z_0, \ldots, z_n; \gamma) \), the tagged simplex

\[
T_R := (z_n, z_1, \ldots, z_{\gamma}, z_{n-1}, z_{n-2}, \ldots, z_{\gamma+1}, z_0; \gamma)
\]

with \( \text{dom}(T_R) = \text{dom}(T) \) has the same children as \( T \). Two tagged simplices \( T, T' \) are called neighbors, if they share a common \((n-1)\)-dimensional hyper-surface. Two neighboring tagged simplices \( T \) and \( T' \) are called \textit{reflected neighbors}, if the ordered sequence of vertices of either \( T \) or \( T_R \) coincides with that of \( T' \) on all but one position; for graphical illustrations cf. [5].

We will impose the following condition on \( \mathcal{T}_0 \).

**Definition 2.1 (Matching condition).** All simplices in \( \mathcal{T}_0 \) are of the same type \( \gamma \). Any two neighboring tagged simplices \( T = (T_0, \ldots, y_n; \gamma) \) and \( T' = (0, \ldots, z_n; \gamma) \) in \( \mathcal{T}_0 \) satisfy the following conditions.
(a) If \( \text{conv}\{y_0, y_n\} \subseteq T \cap T' \) or \( \text{conv}\{z_0, z_n\} \subseteq T \cap T' \), then \( T \) and \( T' \) are reflected neighbors.
(b) If \( \text{conv}\{y_0, y_n\} \subseteq T \cap T' \neq \emptyset \) and \( \text{conv}\{z_0, z_n\} \subseteq T \cap T' \), then any two neighboring children of \( T \) and \( T' \) are reflected neighbors.

The matching condition guarantees that all uniform refinements of \( \mathcal{T}_0 \) are conforming [5, Theorem 4.3], and it is actually needed for this property to hold. For completeness, with a uniform refinement of \( \mathcal{T}_0 \) we mean a descendant of \( \mathcal{T}_0 \) in which all simplices have the same level.

## 3 Refinements

We equip \( \mathcal{T} \) with a partial ordering by defining, for \( \mathcal{T}, \mathcal{T}' \in \mathcal{T}, \mathcal{T} \leq \mathcal{T}' \) when \( \mathcal{T}' \) is a refinement of \( \mathcal{T} \). With this partial ordering, \((\mathcal{T}, \leq)\) is a \textit{lattice}, i.e., for any \( \mathcal{T}, \mathcal{T}' \in \mathcal{T} \), the smallest common refinement \( \mathcal{T} \lor \mathcal{T}' \) and greatest common coarsening \( \mathcal{T} \land \mathcal{T}' \) in \( \mathcal{T} \) are well-defined. A characterization of both these partitions is given in the following remark.

**Remark 3.1.** For \( \mathcal{T}, \mathcal{T}' \in \mathcal{T}, \mathcal{T} \in \mathcal{T} \) and \( \mathcal{T}' \in \mathcal{T}' \) with \( T \subseteq T' \), it holds that \( T' \in \mathcal{T} \land \mathcal{T}' \) and \( T' \in \mathcal{T} \lor \mathcal{T}' \), see, e.g., [4, Lemma 4.3].

For \( \mathcal{T} \in \mathcal{T} \), and a set \( M \subseteq \mathcal{T} \) (the set of simplices ‘marked for refinement’), let

\[
\mathcal{T}' := \text{refine}(\mathcal{T}, M)
\]

denote the \textit{smallest} partition in \( \mathcal{T} \) with \( T \leq T' \) and \( M \cap T' = \emptyset \). The uniform refinement \( \mathcal{T} \) of \( \mathcal{T}_0 \) consisting of all simplices with level equal to \( 1 + \max_{T \in \mathcal{T}} \ell(T) \) satisfies \( \mathcal{T} \leq \mathcal{T} \) and \( \mathcal{T} \cap \mathcal{T} = \emptyset \). Consequently, \( \mathcal{T}' \) is well-defined as the greatest common coarsening of the finite, non-empty set \( \{ \mathcal{T} \in \mathcal{T} : M \cap \mathcal{T} = \emptyset, \mathcal{T} \leq \mathcal{T} \leq \mathcal{T} \} \).
The following result was proved in [5, Theorems 5.1–5.2].

Lemma 3.2. Let $T \in \mathcal{T} \in \mathcal{T}$ and $\mathcal{T}' := \text{refine}(\mathcal{T}, \{T\})$. If $T' \in \mathcal{T}'$ is newly created by the call $\text{refine}(\mathcal{T}, \{T\})$, i.e., $T' \notin \mathcal{T} \setminus \{T\}$, then

(a) $\ell(T') \leq \ell(T) + 1$.
(b) $\text{dist}(T', T) \leq 2^{-\ell(T')/n}$.

We are ready to show that for $T \in \mathcal{T} \in \mathcal{T}$, the difference in levels of any $K' \in \text{refine}(\mathcal{T}, \{T\})$ and its ancestor $K \in \mathcal{T}$ is uniformly bounded.

Theorem 3.3. Let $T \in \mathcal{T} \in \mathcal{T}$ and $\mathcal{T}' = \text{refine}(\mathcal{T}, \{T\})$. Let $K \in \mathcal{T}$ and $K' \in \mathcal{T}'$ with $K' \subset K$. Then it holds that

$$\ell(K') - \ell(K) \leq 1.$$ 

Proof. If $\ell(K') = \ell(K)$, the assertion is trivially valid. Hence, assume that $\ell(K) + 1 \leq \ell(K')$, i.e., $K'$ is newly created by the call. Recall the constant $C$ from Lemma 1.1.

Case 1. If $\ell(T) \leq \ell(K) + C$, then by Lemma 3.2 (a), it holds that $\ell(K') \leq \ell(T) + 1 \leq \ell(K) + C + 1$.

Case 2. If $\ell(T) > \ell(K) + C$, then Lemma 1.1 (b) implies that $\text{dist}(T, K) \geq 2^{-\ell(K)/n}$, whence

$$\text{dist}(T, K') \geq 2^{-\ell(K)/n}.$$

On the other hand, Lemma 3.2 (b) states that

$$\text{dist}(K', T) \leq 2^{-\ell(K)/n}.$$

The foregoing two inequalities imply

$$2^{-\ell(K)/n} \leq 2^{-\ell(K')/n},$$

and so $\ell(K') - \ell(K) \leq 1$. \hfill $\square$

Remark 3.4. In dimension $n = 2$, given $\mathcal{T} \in \mathcal{T}$, the triangulation $\mathcal{T}'$ defined by replacing each $T \in \mathcal{T}$ by its four grandchildren is conforming and so belongs to $\mathcal{T}$. We conclude that for any $T \in \mathcal{T}$, it holds that $\text{refine}(\mathcal{T}, \{T\}) \leq \mathcal{T}'$ giving an easy proof of Theorem 3.3 in this case. Moreover, it yields the additional information that this theorem is valid in this situation with $\ell(K') - \ell(K) \leq 2$.

This argument does not apply in $n > 2$ dimensions. Replacing any $T \in \mathcal{T} \in \mathcal{T}$ by its level $n$-descendants generally does not yield a conforming partition. Indeed, already for $n = 3$, in the partition formed by the level 3 descendants of a tagged tetrahedron $T$ of type 0 or 1, all the edges of $T$ have been cut exactly once, but for a tagged tetrahedron $T$ of type 2, this partition still contains one of the original edges.

The following corollary generalizes Theorem 3.3 to the case that $\text{refine}$ is called with a set of marked elements.

Corollary 3.5. Let $\mathcal{M} \subset \mathcal{T} \in \mathcal{T}$ and $\mathcal{T}' = \text{refine}(\mathcal{T}, \mathcal{M})$. Let $K \in \mathcal{T}$ and $K' \in \mathcal{T}'$ with $K' \subset K$. Then it holds that

$$\ell(K') - \ell(K) \leq 1.$$ 

Proof. It holds that

$$\mathcal{T}' = \bigvee_{T \in \mathcal{M}} \text{refine}(\mathcal{T}, \{T\}),$$

i.e., $\mathcal{T}'$ is the smallest common refinement of the triangulations $\text{refine}(\mathcal{T}, \{T\})$ for $T \in \mathcal{M}$. From Remark 3.1, we infer that for any $K' \in \mathcal{T}'$, there exists a $T \in \mathcal{M}$ with $K' \in \text{refine}(\mathcal{M}, \{T\})$. Thus, Theorem 3.3 proves the assertion. \hfill $\square$

Remark 3.6. Corollary 3.5 accomplishes the proof of [2, Lemma 4.2]. It is furthermore required in [1, p.1201] for the constant $C_{\text{con}}$ in equation (2.8) of [1] to be finite.

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References


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